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the spine in a topological book embedding of a graph

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Abstract

In a topological book embedding of a graph, the graph is drawn in a topological book by placing the vertices along the spine of the book and drawing the edges in the pages; edges are allowed to cross the spine. Earlier results show that every graph having n vertices and m edges can be embedded into a 3-page book with at most $\mathrm{O}(m \log n)$ edge-crossings over the spine. This paper presents lower bounds on the number of edge-crossings over the spine for a variety of graphs. These bounds show that the upper bound $\mathrm{O}(m \log n)$ is essentially best possible.

Keywords: Book embedding; Topological book embedding; Edge crossing

1. Introduction

We consider only undirected graphs without loops or multiple edges. A *k-page book* consists of a line segment called the *spine* and k rectangular regions called the *pages* which contain the spine as their common boundary. A *book embedding* of a graph is a drawing of the graph on a book such that the vertices are placed on the spine and the edges do not cross each other.

Usually, the book embeddings are restricted so that no edge can cross over the spine of the book, i.e., each edge must be drawn in just one of the pages. A book embedding with this restriction is a *combinatorial book embedding*. The *pagenumber* of a graph is the minimum number of pages in which the graph has a combinatorial book embedding. The pagenumbers of graphs are studied in several papers; see [3,5,8,9]. For example, the pagenumber of the complete graph K_n is known to be $\lceil n/2 \rceil$ if $n \neq 3$.

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On the other hand, in a *topological book embedding*, edges are allowed to cross the spine. In contrast with combinatorial book embeddings, every graph has a topological book embedding into a 3-page book [1,2]. For this type of book embedding, the research interest is to find the bounds for the number of crossing points of edges over the spine [6]. Enomoto and Miyauchi [4] prove that every graph can be embedded into a 3-page book so that each edge crosses the spine at $O(\log n)$ points, where n is the number of vertices of the graph. Consequently, every graph having n vertices and m edges can be embedded into a 3-page book with $O(m \log n)$ edge-crossings over the spine.

In this paper, we prove that this bound is essentially the best possible. More precisely, if G_n $(n=1,2,\ldots)$ is a sequence of graphs with $|V(G_n)|=n$ and $|E(G_n)|=\Omega(n^{1+r})$, where r is a positive constant, then a topological embedding of G_n into a 3-page book requires $\Omega(n^{1+r}\log n)$ edge-crossings over the spine. Moreover, we also present analogous results for the topological embeddings into a book consisting of four or more pages.

2. Key lemma and main results

The key lemma to obtain a lower bound for the number of edge-crossings over the spine is the following, which we shall prove in the next section.

Lemma 2.1. *If a graph G can be embedded in a 3-page book so that each edge crosses at most $k-1$ points on the spine, then $|E(G)| < \frac{4}{3}10^k|V(G)|$.* □

Using this lemma, we can prove the following theorem.

Theorem 2.2. *There exist a constant $c_0 > 0$ and an integer n_0 such that if $n \geqslant n_0$, then for any topological embedding of the complete graph K_n into a 3-page book, there must exist $c_0 n^2 \log n$ points at which the edges of K_n cross over the spine.*

Proof. Let ε and ε' be any constants satisfying $0 < \varepsilon' < \varepsilon < 1$. Choose n_0 to be an integer such that, whenever $n \geqslant n_0$, there exists an integer k (a function of n) satisfying

$$(1-\varepsilon)\frac{\log n}{\log 10} \leqslant k \leqslant (1-\varepsilon')\frac{\log n}{\log 10}.$$

Then by Lemma 2.1, for any embedding of K_n into a 3-page book, at least $\binom{n}{2} - \frac{4}{3}10^k n$ edges cross the spine at least k times. Hence, the total number of edge-crossings over the spine is at least

$$\left(\binom{n}{2} - \frac{4}{3}10^k n\right)k \geqslant \left(\binom{n}{2} - \frac{4}{3}n^{2-\varepsilon'}\right)(1-\varepsilon)\frac{\log n}{\log 10}$$
$$= \left(\frac{1-\varepsilon}{2\log 10} - o(1)\right)n^2 \log n.$$

If we choose c_0 to be slightly smaller than $1/2\log 10$, then the theorem follows. □

Theorem 2.2 concerns only complete graphs. However, by the same argument, we can prove a more general result.

Theorem 2.3. *Let G_n $(n=1,2,\ldots)$ be a sequence of graphs such that $|V(G_n)|=n$ and $|E(G_n)|=nf(n)$ where $f(n)\to\infty$ as $n\to\infty$. Then, there exist a constant $c_1>0$ and an integer n_1 such that if $n\geqslant n_1$, then for any embedding of a graph G_n into a 3-page book, there must exist at least $c_1|E(G_n)|\log f(n)$ points at which the edges of G_n cross over the spine.*

Proof. Apply Lemma 2.1 by setting $k\approx(1-\varepsilon)\log f(n)/\log 10$. □

Especially, if G_n has $m=\Omega(n^{1+r})$ edges where r is a positive constant, this theorem implies that any topological embedding into a 3-page book needs at least $\Omega(m\log n)$ crossing points on the spine. This shows that the result in [4] is best possible in a sense.

In [7], it is proved that every graph can be embedded into a $(d+1)$-page book $(d\geqslant 2)$ so that each edge crosses the spine at most $C\log_d n$ times, where C is a positive constant not depending on d. We can generalize Theorems 2.2 and 2.3 to the embeddings of graphs into a $(d+1)$-page book with $d\geqslant 3$. First, the following lemma is a generalization of Lemma 2.1. We leave the proof to the next section.

Lemma 2.4. *If a graph G can be embedded in a $(d+1)$-page book $(d\geqslant 2)$ so that each edge crosses at most $k-1$ points on the spine, then $|E(G)|<[(4d+4)/(5d-1)](5d)^k|V(G)|$.*

Using this lemma, the proof technique of Theorem 2.2 applies to the following theorems, which show the upper bounds of [7] are best possible within constant factors.

Theorem 2.5. *Let d be an integer with $d\geqslant 2$. There exist an absolute constant $c_2>0$ (not depending on d), and an integer $n_2=n_2(d)$ such that if $n\geqslant n_2$, then for any embedding of the complete graph K_n into a $(d+1)$-page book, there must exist $c_2n^2(\log n/\log d)$ points at which the edges of K_n cross over the spine.*

Theorem 2.6. *Let d be an integer with $d\geqslant 2$, and G_n $(n=1,2,\ldots)$ be a sequence of graphs such that $|V(G_n)|=n$ and $|E(G_n)|=nf(n)$ where $f(n)\to\infty$ as $n\to\infty$. Then, there exist a constant $c_3>0$ (not depending on d but depending on f), and an integer $n_3=n_3(d,f)$ such that if $n\geqslant n_3$, then for any embedding of a graph G_n into a $(d+1)$-page book, there must exist at least $c_3|E(G_n)|(\log f(n)/\log d)$ points at which the edges of G_n cross over the spine.*

3. Proof of key lemma

In this section, we prove the key lemmas presented in the previous section. It is easy to see that Lemma 2.1 is the case $d=2$ of Lemma 2.4. So we give the proof of Lemma 2.4 only.

First, we introduce some notation and terminology. In order to treat a topological book embedding combinatorially, we consider a corresponding combinatorial book embedding. Suppose that a topological book embedding of a graph G is given. At each crossing point of edges over the spine, we put a new vertex and subdivide the edge. Then, we obtain a combinatorial book embedding of a subdivision of G. The resulting graph is the *subdivision associated with the topological book embedding of* G.

For convenience, we always identify the spine of a book as the unit interval $[0,1]$ of the real line. For a graph G embedded in a book, let $\varphi : V(G) \to [0,1]$ denote the embedding of the vertices on the spine. Suppose that a combinatorial book embedding of a graph G is given. We give an orientation to each edge of G. Let $e = xy$ be an edge of G having an orientation from x to y. The *sign* of the edge e is defined as the sign of $\varphi(y) - \varphi(x)$. We say that a combinatorial book embedding of an oriented graph G is *uniform* if for each page of the book, all the edges drawn on the page have the same sign. We also say that a topological book embedding is uniform if the corresponding combinatorial book embedding of the subdivision derived from that topological book embedding is uniform.

In order to prove Lemma 2.4, we first prove the following three lemmas, in which we deal mainly with uniformly embedded oriented bipartite graphs. A bipartite graph with partite sets X and Y and edge set E is denoted by $(X, Y; E)$. We also assume that all edges are oriented from X to Y. A *star* is a complete bipartite graph in which one of the partite sets consists of one vertex called the *center*; the other vertices are called *leaves*.

Lemma 3.1. *If a bipartite graph* $G = (X, Y; E)$ *is uniformly embedded in a* 1*-page book, then* G *does not contain a cycle, and consequently* $|E| \leqslant |X| + |Y| - 1$.

Proof. Without loss of generality, we may assume that the sign of every edge is plus, i.e., $\varphi(x) < \varphi(y)$ holds for any edge xy with $x \in X$ and $y \in Y$. Suppose that G has a cycle $C = x_1y_1x_2y_2 \cdots x_ly_lx_1$ where $x_i \in X$ and $y_i \in Y$ $(1 \leqslant i \leqslant l)$. We may assume that y_1 is the leftmost vertex in $V(C) \cap Y$. We may also assume that $\varphi(x_1) < \varphi(x_2)$. Then, the vertices x_1, x_2, y_1 and y_2 are lying on the spine in this order, and hence the edges x_1y_1 and x_2y_2 cross each other. This is a contradiction. Thus, G can not contain a cycle, and the result follows. □

Lemma 3.2. *Suppose that a bipartite graph* $G = (X, Y; E)$ *is a vertex-disjoint union of* n *stars, where* X *is the set of centers and* Y *is the set of leaves. If* G *is uniformly embedded in a* 1*-page book, then the spine* $[0,1]$ *can be decomposed into* t *intervals* $I_1, I_2, \ldots, I_t$, *where* $t \leqslant 2n - 1$, *such that for each* i $(1 \leqslant i \leqslant t)$ *the vertices of* Y *lying on* I_i *belong to the same component of* G.

Proof. We may assume that the sign of every edge is plus, i.e., $\varphi(x) < \varphi(y)$ holds for any edge xy with $x \in X$ and $y \in Y$. Let $x_1, \ldots, x_n$ be the vertices of X so that $\varphi(x_1) < \cdots < \varphi(x_n)$. Note that the interval $[0, \varphi(x_1)]$ contains no vertex of Y by assumption. Define the graph G^* to be obtained from G by the following operations:

(1) For each i $(1 \leqslant i \leqslant n)$, contract the vertices of Y between x_i and x_{i+1} into y_i^*, where x_{n+1} is considered as the rightmost point of the spine.

(2) Replace each resulting multiple edge with a single edge.

Then, G^* is a bipartite graph uniformly embedded in a 1-page book, and has at most $2n$ vertices. Hence by Lemma 3.1, G^* has at most $2n-1$ edges. For each edge $e^* = x_j y_i^*$ of G^*, the vertices of

$$\{y \in Y | x_j y \in E \text{ is mapped to } e^* \text{ by the contraction}\}$$

appear consecutively on the spine. This implies that if y_i^* is incident with d_i edges in G^*, then the interval $[\varphi(x_i), \varphi(x_{i+1})]$ can be decomposed into d_i intervals such that the vertices of Y lying on each interval are adjacent to the same vertex in X. In total, the spine can be decomposed into $\sum_{i=1}^{n} d_i = |E(G^*)| \leqslant 2n-1$ intervals satisfying the desired condition. $\square$

Lemma 3.3. *Let $G=(X,Y;E)$ be a bipartite graph. Let H be a subdivision of G such that each edge $e=xy \in E$ $(x \in X, y \in Y)$ corresponds to a path*

$$P(e) = v_0^{(e)} v_1^{(e)} \cdots v_l^{(e)} \qquad (x = v_0^{(e)}, y = v_l^{(e)})$$

of length l in H. Suppose that H is uniformly embedded in an l-page book so that for any $e \in E$ and $1 \leqslant i \leqslant l$, $v_{i-1}^{(e)} v_i^{(e)}$ lies on the ith page. Then, $|E| < (\frac{5}{2})^{l-1}(|X|+|Y|)$.

Proof. Induction on l. In the case $l=1$, the conclusion follows from Lemma 3.1. Suppose $l \geqslant 2$. We may assume that $|X| \leqslant |Y|$. Let $n=|X|$. Let H' be the subgraph of H induced by the edges lying on the first page. Note that the graph H' is a disjoint union of stars in which the centers are the vertices of X and the leaves are the vertices $v_1^{(e)}$, $e \in E$. Then by Lemma 3.2, the spine $[0,1]$ can be decomposed into at most $2n-1$ intervals $I_1, I_2, \ldots, I_t$ $(t \leqslant 2n-1)$ such that for each j $(1 \leqslant j \leqslant t)$, every vertex $v_1^{(e)}$ lying on I_j is adjacent to the same vertex $x_j \in X$. (Here, x_j $(1 \leqslant j \leqslant t)$ need not be distinct.)

Now, we consider the edges on the second page. Define

$$E_1 = \{e \in E \mid \varphi(v_1^{(e)}) \in I_j \text{ and } \varphi(v_2^{(e)}) \in I_j \text{ for some } j\},$$
$$E_2 = E - E_1.$$

Let G_1 and G_2 denote the subgraphs of G induced by E_1 and E_2, respectively. We shall bound the sizes of E_1 and E_2 separately.

Claim 1. $|E_1| < (\frac{5}{2})^{l-2}(|X|+|Y|)$.

Proof of Claim 1. From the given embedding of the paths $P(e)$ with $e \in E_1$ into the l-page book, we shall construct a uniform embedding of G_1 into an $(l-1)$-page book as follows. First, delete all edges drawn in the first and the second pages, erase all vertices $v_1^{(e)}$ with $e \in E_1$. Next, take a new page, and for each $e \in E_1$ draw an edge

$v_0^{(e)}v_2^{(e)}$ in the new page. Here, if $v_0^{(e)}$ and $v_2^{(e)}$ are lying on a same interval I_j, then it may be necessary to move $v_0^{(e)}$ to one end of I_j to keep all edges of the form $v_0^{(e)}v_2^{(e)}$ in the proper direction. Then, we obtain a uniform embedding of G_1 into an $(l-1)$-page book, as desired. By the induction hypothesis, $|E(G_1)|=|E_1| < (\frac{5}{2})^{l-2}(|X|+|Y|)$ holds.

Claim 2. $|E_2| < (\frac{5}{2})^{l-2}(2|X|+|Y|-1)$.

Proof of Claim 2. Consider the given embedding of the paths $P(e)$ with $e\in E_2$ into the l-page book, induced by the embedding of H. Delete all vertices of X and all edges drawn in the first page. Now, for each j $(1\leqslant j\leqslant t)$, contract the vertices $v_1^{(e)}$ lying on I_j into a single vertex z_j, and put it on I_j. Define $Z=\{z_1,z_2,\dots,z_t\}$. Let H_2 be the resulting graph. Since the original bipartite graph G has no multiple edges and all $v_1^{(e)}$'s in I_j are adjacent to the same vertex x_j, they must lead to distinct y's. Also, since every edge of the form $v_1^{(e)}v_2^{(e)}$ in the second page did not stay within the interval, these contractions preserve the signs of all edges in the second page. Hence H_2 is uniformly embedded in an $(l-1)$-page book.

Consequently, H_2 is a subdivision of a bipartite graph, say G_2^*, having the partite sets Z and Y. By the induction hypothesis, $|E(G_2^*)|=|E_2| < (\frac{5}{2})^{l-2}(|Z|+|Y|) \leqslant (\frac{5}{2})^{l-2}(2|X|-1+|Y|)$ holds.

By Claim 1 and Claim 2,

$$\begin{aligned}
|E| &= |E_1|+|E_2| \\
&< (\tfrac{5}{2})^{l-2}(|X|+|Y|)+(\tfrac{5}{2})^{l-2}(2|X|+|Y|-1) \\
&\leqslant (\tfrac{5}{2})^{l-2}(|X|+|Y|)+(\tfrac{5}{2})^{l-2}(\tfrac{3}{2}|X|+\tfrac{3}{2}|Y|-1) \\
&< (\tfrac{5}{2})^{l-1}(|X|+|Y|).
\end{aligned}$$

This completes the proof of Lemma 3.3. □

Now we can prove Lemma 2.4.

Proof of Lemma 2.4. Suppose that G is embedded in a $(d+1)$-page book such that each edge crosses at most $k-1$ points on the spine. We give an arbitrary orientation to each edge of G. Let H be the subdivision associated with the embedding of G. For an edge $e=xy$ of G oriented from x to y, we denote the corresponding path in H by

$$P(e)=v_0^{(e)}v_1^{(e)}\cdots v_l^{(e)} \qquad (x=v_0^{(e)},\, y=v_l^{(e)})$$

Then by the assumption, the length l of $P(e)$ is at most k for any e.

We fix an integer l with $1\leqslant l\leqslant k$, and a sequence of l pages $p_1p_2\cdots p_l$ with $p_i\neq p_{i+1}$ $(1\leqslant i\leqslant l-1)$. For each page p_i, we also fix a sign (plus/minus) s_i. Let G' be the subgraph of G induced by the edges $e=xy$ such that

(1) the length of $P(e)$ is exactly l,
(2) $P(e)$ passes the pages p_1, $p_2,\dots,p_l$ in this order, and
(3) the sign of the edge $v_{i-1}^{(e)}v_i^{(e)}$ is s_i $(1\leqslant i\leqslant l)$.

Now we shall bound the number of edges in G'. If $l=1$, then since G' is an outerplanar graph, we have $|E(G')| < 2|V(G)|$.

Suppose that $l \geqslant 2$. Take a new l-page book, and consider the same mapping φ of the vertices of H into the spine. First, we draw every edge $v_{i-1}^{(e)}v_i^{(e)}$ onto the ith page. Next, split each vertex x of G' into x' and x'' so that each edge xy of G' joins x' and y''. Here the order of x' and x'' is irrelevant, since x'' is isolated in the first page and x' is isolated in the last page. Then, we obtain a bipartite graph and its topological book embedding satisfying the assumption of Lemma 3.3. Hence, $|E(G')| < 2(\frac{5}{2})^{l-1}|V(G)|$ holds.

For a fixed l, there are at most $(d+1)d^{l-1}$ sequences of l-pages, and there are 2^l possibilities for their signs. Also l varies from 1 to k. By summing up $|E(G')|$ for all these possibilities,

$$|E(G)| < \sum_{l=1}^{k} 2(\tfrac{5}{2})^{l-1}|V(G)| \cdot (d+1)d^{l-1} \cdot 2^l = 4(d+1)|V(G)|\sum_{l=1}^{k}(5d)^{l-1}$$
$$< \frac{4d+4}{5d-1}(5d)^k|V(G)|.$$

This completes the proof of the lemma. $\square$

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